**Functions, Limits and Continuity**

**Function of a complex variable:** If a complex variable  depends on another complex variable in such a way that each value of determines exactly one value of , then  is called a function of  and it is written as,



where is independent variable and  is dependent variable.

Every complex function can be expressed as,



where  and  are two real functions of real variables  and .

Example: (1). 

(2). 

**Single-valued function:** A function  is single valued function if there exists only one value of  for each value of .

Example:  is a single-valued function.

**Multiple-valued function:** A function  is many- valued function if there exists more than one value of  for each value of . Actually, the many-valued function is the collection of single-valued function.

Example:  is a multiple-valued function.

**Inverse function:** If be a complex function of , then  is also a complex function of . The function  is often called the inverse function corresponding to .

**Neighbourhoods:** The neighbourhood of a point is the set of all points such that  where is any positive number. The deleted  neighbourhood of  is a neighbourhood of in which the point is omitted, i.e. .

**Limit:** Let  be defined and single-valued in a neighbourhood of . The number  is called limit of  as  tends to  and write  if for any positive number  (however small) we can find some positive number  (usually depending on ) such that  whenever .

Alternatively, the number  is called limit of if  approaches to  as approaches to .

**Theorem-01:** If  exists, then prove that it must be unique.

**Proof:** We must show that if  and  , then .

By hypothesis, for any given , there exists  such that

 when 

and  when .

Now 









This means  is less than any positive number  (however small) and it must be equal to zero.

i.e. 



Thus if  exists, then it must be unique. **(Proved)**

**Continuity:** Let  be defined and single-valued in a neighbourhood of  and is the functional value of it at . The function is said to be continuous at , if for any , we can find  such that  whenever .

Alternatively, the function is said to be continuous at if the following conditions are satisfied:

(1).  must exist

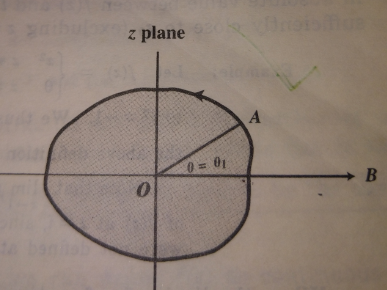
(2).  must exist, i.e. is defined at 

(3). .

A function  is said to be continuous in a region if it is continuous at all points of the region.

**Uniform continuity:** The function  is uniformly continuous in a region if for any , we can find  such that  whenever  where and  are any two points of the region.

**Branch points and Branch lines:** Suppose we have the function  and we allow to make a complete circuit (counterclockwise) around the origin starting from point . We have ,  so that at ,  and . After a complete circuit back to ,  and . Thus we have not achieved the same value of  with which we started. However, by making a second complete circuit back to , i.e. ,  and we then do obtain the same value of  with which we started. We can describe the above by stating that if  we are on one branch of the multiple-valued function , while if  we are on another branch of the function.



It is clear that each branch of the function is single-valued. In order to keep the function single –valued, we set up an artificial barrier such as OB where B is at infinity which we agree not to cross. This barrier is called a branch line or branch cut and point O is called a branch point. It should be noted that a circuit around any point other than  does not lead to different values; thus  is the only finite branch point.

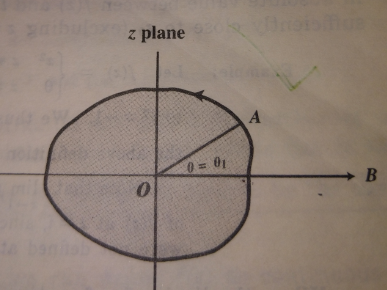
**Problems**

**Problem-01:** Prove that  has a branch point at .

**Solution:** We have 







Suppose we start from  at which , 



After making a complete circuit in the counter clockwise direction and back to , we have

, 

.

We have not achieved the same value with which we have started.

Thus, we have another branch of  and so  is a branch point. **(Proved)**

**Problem-02:** Prove that  does not exist.

**Solution:** Let . Then .



Taking limit along the real axis , we have

.

Again, taking limit along the imaginary axis , we have

.

The above two limits are not equal, that is, the limit depends on manner in which .

Hence  does not exist. **(Proved)**

**Problem-03:** If , then prove that .

**Solution:** We must show that for any given , we can find (depending on ) such that

 whenever .

If , then  implies that









Take  as  or , whichever is smaller.

Then we have,

 whenever .

Hence the required result is proved.

**Problem-04:** Prove that , where , is discontinuous at .

**Solution:** We have 

Now, 

and .

Since , so is discontinuous at  if . **(Proved)**

**Problem-05:** Prove that is uniformly continuous in the region .

**Solution:** We must show that for any given , we can find (depending only on  but not on the any particular point  of the region ) such that

 when .

If and  are any points in , then







Thus if , it follows that



Choosing , we see that

.

Hence  is uniformly continuous in the region. **(Proved)**

**Problem-06:** Prove that is not uniformly continuous in the region .

**Solution:** Let  and  be any two points of the region such that

.

Then 





.

This can be made larger than any positive number by choosing sufficiently close to .

Hence the function can not be uniformly continuous in the region. **(Proved)**